

### Formula sheet

**Co-occurrence matrix**  $g(i, j) = \{\text{no. of pixel pairs with grey levels } (z_i, z_j) \text{ satisfying predicate } Q\}$ ,  $1 \leq i, j \leq L$

**Convolution, 2-D discrete**  $(f \star h)(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x-m, y-n)$ ,  
for  $x = 0, 1, 2, \dots, M-1, y = 0, 1, 2, \dots, N-1$

**Convolution Theorem, 2-D discrete**  $\mathcal{F}\{f \star h\}(u, v) = F(u, v) H(u, v)$

**Distance measures** Euclidean:  $D_e(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$ , City-block:  $D_4(p, q) = |p_1 - q_1| + |p_2 - q_2|$ , Chessboard:  $D_8(p, q) = \max(|p_1 - q_1|, |p_2 - q_2|)$

**Entropy, source**  $H = -\sum_{j=1}^J P(a_j) \log P(a_j)$

**Entropy, estimated** for  $L$ -level image:  $\tilde{H} = -\sum_{k=0}^{L-1} p_r(r_k) \log_2 p_r(r_k)$

**Error, root-mean square**  $e_{\text{rms}} = \left[ \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} (\hat{f}(x, y) - f(x, y))^2 \right]^{\frac{1}{2}}$

**Exponentials**  $e^{ix} = \cos x + i \sin x$ ;  $\cos x = (e^{ix} + e^{-ix})/2$ ;  $\sin x = (e^{ix} - e^{-ix})/2i$

**Filter, inverse**  $\hat{f} = f + \mathbf{H}^{-1} \mathbf{n}$ ,  $\hat{F}(u, v) = F(u, v) + \frac{N(u, v)}{H(u, v)}$

**Filter, parametric Wiener**  $\hat{f} = (\mathbf{H}^t \mathbf{H} + K \mathbf{I})^{-1} \mathbf{H}^t \mathbf{g}$ ,  $\hat{F}(u, v) = \left[ \frac{H^*(u, v)}{|H(u, v)|^2 + K} \right] G(u, v)$

**Fourier series** of signal with period  $T$ :  $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2\pi n}{T} t}$ , with Fourier coefficients:  
 $c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-i \frac{2\pi n}{T} t} dt$ ,  $n = 0, \pm 1, \pm 2, \dots$

**Fourier transform 1-D (continuous)**  $F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\mu t} dt$

**Fourier transform 1-D, inverse (continuous)**  $f(t) = \int_{-\infty}^{\infty} F(\mu) e^{i2\pi\mu t} d\mu$

**Fourier Transform, 2-D Discrete**  $F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-i2\pi(xu/M + vy/N)}$   
for  $u = 0, 1, 2, \dots, M-1, v = 0, 1, 2, \dots, N-1$

**Fourier Transform, 2-D Inverse Discrete**  $f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{i2\pi(xu/M + vy/N)}$   
for  $x = 0, 1, 2, \dots, M-1, y = 0, 1, \dots, N-1$

**Fourier spectrum** Fourier transform of  $f(x, y)$ :  $F(u, v) = R(u, v) + i I(u, v)$ , Fourier spectrum:  $|F(u, v)| = \sqrt{R^2(u, v) + I^2(u, v)}$ , phase angle:  $\phi(u, v) = \arctan\left(\frac{I(u, v)}{R(u, v)}\right)$

**Gaussian function** mean  $\mu$ , variance  $\sigma^2$ :  $G_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

**Gradient**  $\nabla f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$

**Histogram**  $h(m) = \#\{(x, y) \in D : f(x, y) = m\}$ . Cumulative histogram:  $P(\ell) = \sum_{m=0}^{\ell} h(m)$

**Impulse, discrete**  $\delta(0) = 1, \delta(x) = 0$  for  $x \in \mathbb{N} \setminus \{0\}$

**Impulse, continuous**  $\delta(\infty) = 1, \delta(x) = 0$  for  $x \neq 0$ , with  $\int_{-\infty}^{\infty} f(t) \delta(t - t_0) dt = f(t_0)$

**Impulse train**  $s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$ , with Fourier transform  $S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$

**Laplacian**  $\nabla^2 f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

**Laplacian-of-Gaussian**  $\nabla^2 G_\sigma(x, y) = -\frac{2}{\pi\sigma^4} \left( 1 - \frac{r^2}{2\sigma^2} \right) e^{-r^2/2\sigma^2}$  ( $r^2 = x^2 + y^2$ )

**Morphology**

**Dilation**  $\delta_A(X) = X \oplus A = \bigcup_{a \in A} X_a = \bigcup_{x \in X} A_x = \{h \in E : \check{A}_h \cap X \neq \emptyset\}$ ,

where  $X_h = \{x + h : x \in X\}$ ,  $h \in E$  and  $A = \{-a : a \in A\}$

**Erosion**  $\varepsilon_A(X) = X \ominus A = \bigcap_{a \in A} X_{-a} = \{h \in E : A_h \subseteq X\}$

**Opening**  $\gamma_A(X) = X \circ A := (X \ominus A) \oplus A = \delta_A \varepsilon_A(X)$

**Closing**  $\phi_A(X) = X \bullet A := (X \oplus A) \ominus A = \varepsilon_A \delta_A(X)$

**Hit-or-miss transform**  $X \otimes (B_1, B_2) = (X \ominus B_1) \cap (X^c \ominus B_2)$

**Thinning**  $X \otimes B = X \setminus (X \otimes B)$ , **Thickening**  $X \odot B = X \cup (X \otimes B)$

**Morphological reconstruction** Marker  $F$ , mask  $G$ , structuring element  $B$ :

$X_0 = F$ ,  $X_k = (X_{k-1} \oplus B) \cap G$ ,  $k = 1, 2, 3, \dots$

**Morphological skeleton** Image  $X$ , structuring element  $B$ :  $SK(X) = \bigcup_{n=0}^N S_n(X)$ ,

$S_n(X) = X \ominus_n B \setminus (X \ominus_{n+1} B) \odot B$ ,  $S_0(X) = X$ , with  $N$  the largest integer such that  $S_N(X) \neq \emptyset$

**Grey value dilation**  $(f \oplus b)(x, y) = \max_{(s,t) \in B} [f(x-s, y-t) + b(s, t)]$

**Grey value erosion**  $(f \ominus b)(x, y) = \min_{(s,t) \in B} [f(x+s, y+t) - b(s, t)]$

**Grey value opening**  $f \circ b = (f \ominus b) \oplus b$

**Grey value closing**  $f \bullet b = (f \oplus b) \ominus b$

**Morphological gradient**  $g = (f \oplus b) - (f \ominus b)$

**Top-hat filter**  $T_{\text{hat}} = f - (f \circ b)$ , **Bottom-hat filter**  $B_{\text{hat}} = (f \bullet b) - f$

**Sampling** of continuous function  $f(t)$ :  $\tilde{f}(t) = f(t) s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t) \delta(t - n\Delta T)$ .

Fourier transform of sampled function:  $\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta T})$

**Sampling theorem** Signal  $f(t)$ , bandwidth  $\mu_{\text{max}}$ : If  $\frac{1}{\Delta T} \geq 2\mu_{\text{max}}$ ,  $f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T) \text{sinc} \left[ \frac{t-n\Delta T}{n\Delta T} \right]$ .

**Sampling: downsampling** by a factor of 2:  $\downarrow_2(a_0, a_1, a_2, \dots, a_{2N-1}) = (a_0, a_2, a_4, \dots, a_{2N-2})$

**Sampling: upsampling** by a factor of 2:  $\uparrow_2(a_0, a_1, a_2, \dots, a_{N-1}) = (a_0, 0, a_1, 0, a_2, 0, \dots, a_{N-1}, 0)$

**Set, circularity ratio**  $R_c = \frac{4\pi A}{P^2}$  of set with area  $A$ , perimeter  $P$

**Set, diameter**  $\text{Diam}(B) = \max_{i,j} [D(p_i, p_j)]$  with  $p_i, p_j$  on the boundary  $B$  and  $D$  a distance measure

**Sinc function**  $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$  when  $x \neq 0$ , and  $\text{sinc}(0) = 1$

**Spatial moments** of an  $M \times N$  image  $f(x, y)$ :  $m_{pq} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} x^p y^q f(x, y)$ ,  $p, q = 0, 1, 2, \dots$

**Statistical moments** of distribution  $p(i)$ :  $\mu_n = \sum_{i=0}^{L-1} (i-m)^n p(i)$ ,  $m = \sum_{i=0}^{L-1} i p(i)$

**Signal-to-noise ratio, mean-square**  $\text{SNR}_{\text{rms}} = \frac{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \hat{f}(x, y)^2}{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} (\hat{f}(x, y) - f(x, y))^2}$

**Wavelet decomposition** with low pass filter  $h_\phi$ , band pass filter  $h_\psi$ . For  $j = 1, \dots, J$ :

Approximation:  $c_j = \text{H}c_{j-1} = \downarrow_2(h_\phi * c_{j-1})$ ; Detail:  $d_j = \text{G}c_{j-1} = \downarrow_2(h_\psi * c_{j-1})$

**Wavelet reconstruction** with low pass filter  $\tilde{h}_\phi$ , band pass filter  $\tilde{h}_\psi$ . For  $j = J, J-1, \dots, 1$ :

$c_{j-1} = \tilde{h}_\phi * (\uparrow_2 c_j) + \tilde{h}_\psi * (\uparrow_2 d_j)$

**Wavelet, Haar basis**  $h_\phi = \frac{1}{\sqrt{2}}(1, 1)$ ,  $h_\psi = \frac{1}{\sqrt{2}}(1, -1)$ ,  $\tilde{h}_\phi = \frac{1}{\sqrt{2}}(1, 1)$ ,  $\tilde{h}_\psi = \frac{1}{\sqrt{2}}(1, -1)$